The arithmetic Hodge index theorem for adelic line bundles I: number fields

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1 Introduction

The Hodge index theorem for divisors on arithmetic surfaces proved by Faltings [Fal] in 1984 and Hirijac [Hr] in 1985 is one of the fundamental results in Arakelov theory. For example, it was used to prove the the first case of Bogomolov's conjecture for curves in tori by Zhang [Zh1]. In 1996, Moriwaki [Mo1] extended the Hodge index theorem for codimension one cycles on high-dimensional arithmetic varieties, and then confirmed the codimension one case of the arithmetic standard conjecture proposed by Gillet and Soulé in [GS3]. Despite its fundamental importance in number theory and arithmetic geometry, e.g, in Gross–Zagier type formula for cycles on Shimura varieties, the high-codimensional case of the Gillet–Soulé conjecture is still wide open.

The aim of this series of two papers is to prove an *adelic version* of the Hodge index theorem for (still) codimension one cycles on varieties over a finitely generated field K. Namely, these line bundles are equipped with metrics as limits of integral models of the structure morphisms. These matrices naturally appear in algebraic dynamical systems and moduli spaces of varieties. Here we will give two applications:

(1) the uniqueness part of the Calabi–Yau theorem for metrized line bundles over non-archimedean analytic spaces,

(2) a rigidity result of the sets of preperiodic points of polarizable endomorphisms of a projective variety over any field K.

The proof of our results uses Arakelov theory (cf. [Ar, GS1]) and Berkovich analytic spaces (cf. [Be]). In comparison with Moriwaki's proof, one essential difficulty in adelic case is the lack of relative ampleness in projective systems of integral models. Our new ideas for this are a new notion of $\bar{L}$-boundedness and a variation method (inspired by Blocki's work [Bl] in complex geometry).

In this paper, the first one of the series, we prove our Hodge index theorem and the application in (2) assuming K is a number field, and prove the application in (1) in the full generality. In [YZ], the second one of the series, we will treat our Hodge index theorem and the application in (2) in the full generality after introducing a theory of adelic line bundles on varieties over finitely generated fields. We separate the exposition into two papers due to the technicality of our theory over finitely generated fields. In the following, we state the main results of this paper.

1.1 Arithmetic Hodge index theorem

Let us first recall the arithmetic Hodge index theorem for Hermitian line bundles on arithmetic varieties.

Theorem 1.1 ([Fal, Hr, Mo1]). *Let K be a number field, and $\pi : \mathcal{X} \to \operatorname{Spec} O_K$ be a regular arithmetic variety, geometrically connected of relative dimension $n \geq 1$. Let $\overline{\mathcal{M}}$ be a Hermitian line bundle on $\mathcal{X}$, and $\overline{\mathcal{L}}$ be an ample Hermitian line bundle on $\mathcal{X}$. Assume that $\mathcal{M}_K \cdot \mathcal{L}_K^{n-1} = 0$ on the generic fiber $\mathcal{X}_K$. Then the arithmetic intersection number*

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}^{n-1} \leq 0.$$

Moreover, if $\mathcal{L}$ is ample on $\mathcal{X}$ and the metric of $\overline{\mathcal{L}}$ is strictly positive, then the equality holds if and only if $\overline{\mathcal{M}} = \pi^* \overline{\mathcal{M}}_0$ for some Hermitian line bundle $\overline{\mathcal{M}}_0$ on $\operatorname{Spec} O_K$.

The result was due to Faltings [Fal] and Hriljac [Hr] for $n = 1$, and due to Moriwaki [Mo1] for general n.

The main result of this paper is a version of the above result for adelic line bundles. The importance of the Hodge index theorem in the adelic setting will be justified by our applications to the Calabi–Yau theorem and to algebraic dynamics.

To state our theorem, we start with some positivity notions. We refer to Zhang [Zh3] for basic definitions of adelic line bundles, and to Definition 2.1 for positivity of Hermitian line bundles on arithmetic varieties.

Definition 1.2. *Let K be a number field. Let X be a projective variety over K, and $\overline{L}, \overline{M}$ be adelic line bundles on X. We make the following definitions.*

(1) We say that $\overline{L}$ is nef *if the adelic metric of L is a uniform limit of metrics induced by nef Hermitian line bundles on integral models of X.*

(2) We say that $\overline{L}$ is integrable *if it is the difference of two nef adelic line bundles on X.*

(3) We say that $\overline{L}$ is ample *if L is ample, $\overline{L}$ is nef, and $(\overline{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X.*

*(4) We say that $\overline{M}$ is $\overline{L}$-*bounded *if there is an integer $m > 0$ such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are nef.*

The following is the main theorem of this paper.

Theorem 1.3. *Let K be a number field, and $\pi : X \to \operatorname{Spec} K$ be a normal and geometrically connected projective variety of dimension $n \geq 1$. Let $\overline{M}$ be an integrable adelic line bundle on X, and $\overline{L}_1, \cdots, \overline{L}_{n-1}$ be $n-1$ nef line bundles on X where each L_i is big on X. Assume $M \cdot L_1 \cdots L_{n-1} = 0$ on X. Then*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, $\overline{L}_i$ is ample and $\overline{M}$ is $\overline{L}_i$-bounded for each i, then the equality holds if and only if $r\overline{M} = \pi^*\overline{M}_0$ for some adelic line bundle $\overline{M}_0$ on $\operatorname{Spec} K$ *and some integer $r > 0$.*

If M is numerically trivial, we can strengthen the theorem as follows. First, we can remove the condition "L_i is big" in the inequality part of the theorem, by viewing $\overline{L}_i$ as the limit of $\overline{L}_i + \epsilon\overline{A}$ as $\epsilon \to 0$ for some ample $\overline{A}$. Second, we can replace the condition "$\overline{L}_i$ is ample" by "L_i is ample" in the equality part of the theorem, by changing the metric of $\overline{L}_i$ by constant multiples.

As in the classical case, the theorem explains the signature of the intersection pairing on certain space of adelic line bundles. Let W denote the subspace of $\widehat{\operatorname{Pic}}(X)_{\rm int} \otimes_{\mathbb{Z}} \mathbb{Q}$ consisting of elements which are represented by $\mathbb{Q}$-linear combinations of integrable adelic line bundles on X which are $\overline{L}_i$-bounded for all i. Define a pairing on W by

$$\langle \overline{M}_1, \overline{M}_2 \rangle = \overline{M}_1 \cdot \overline{M}_2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

Denote $V = \pi^*\widehat{\operatorname{Pic}}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$, viewed as a subspace of W. Then the theorem implies that the pairing on $V^\perp$ is negative semi-definite, that V is a maximal isotropic subspace of $V^\perp$, and that $V^\perp/V$ is negative definite.

The inequality part of Theorem 1.3 can be eventually reduced to Theorem 1.1 by taking limits (in the case all $\overline{L}_i$ are equal). However, the equality part of the theorem is more profound and significantly more difficult, since it is not a simple limit of its counterpart for Hermitian line bundles on integral models. The following are some new ingredients of our treatment:

(1) We introduce the notion "$\overline{L}$-bounded" to overcome the difficulty that arithmetic ampleness of Hermitian line bundles is not preserved by pullback by morphisms between integral models, and the difficulty that arithmetic ampleness of adelic line bundles is not preserved by perturbations.

(2) We use a variational method to show that, in the case that M is numerically trivial, if $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0$ holds, then it holds after replacing $\overline{L}_i$ by any nef adelic line bundle $\overline{L}_i^0$ with the same underlying line bundle L_i. It gives much flexibility. The variational method is inspired by the idea of Blocki [Bl] on the complex Calabi–Yau theorem.

(3) We only assume that X is normal and projective in order to cover all polarizable algebraic dynamical systems. To overcome the difficulty, we extend the classical Lefschetz theorems for line bundles to the normal case.

Example 1.4. We give some examples to show that the conditions of the theorem are necessary.

(1) The assumption "L_i is big" is necessary for the inequality if $\overline{M}$ is not vertical. In fact, take $n = 2$. Take $\overline{L}_1 = \pi^* \overline{N}$ for any adelic line bundle $\overline{N} \in \widehat{\mathrm{Pic}}(K)$ with $\widehat{\deg}(\overline{N}) = 1$. Note that $L_1 = \mathcal{O}_X$ is trivial and the constraint $M \cdot L_1 = 0$ is automatic. Then the inequality $\overline{M}^2 \cdot \overline{L}_1 \leq 0$ is just $M^2 \leq 0$ for any line bundle M on X. It is not true.

(2) The assumption "$\overline{M}$ is $\overline{L}_i$-bounded for each i" is necessary for the condition of the equality. In fact, take $n = 2$. Take an integral model $\mathcal{X}$ of X. Let $\overline{\mathcal{L}}$ be an ample Hermitian line bundle on $\mathcal{X}$. Let $\alpha : \mathcal{X}' \to \mathcal{X}$ be the blowing-up of a closed point on the $\mathcal{X}$. Let $\overline{\mathcal{M}}$ be the (vertical) line bundle on $\mathcal{X}'$ associated to the exceptional divisor endowed with the trivial hermitian metric $\|1\| = 1$. Then we $\mathcal{M}_K \cdot \mathcal{L}_K = 0$ and $\overline{\mathcal{M}}^2 \cdot \alpha^* \overline{\mathcal{L}} = 0$ by the projection formula. But $\overline{\mathcal{M}}$ is not coming from any line bundle on the base $\mathrm{Spec}\, O_K$. The problem is that $\overline{\mathcal{M}}$ is not $\overline{\mathcal{L}}$-bounded if we convert the objects to the adelic setting.

(3) The assumption "$\overline{L}_i$ is ample" is necessary for the condition of the equality if M is not numerically trivial. In fact, take X to be an abelian variety of dimension n. The multiplication $f = [2]$ defines a polarizable algebraic dynamical system on X. Let N_1, N_2 be two symmetric and ample line bundles on X, which are not proportional. They polarizes f in the sense that $f^* N_i = 4 N_i$ for $i = 1, 2$. Let $\overline{N}_i$ be the f-invariant adelic line bundles extending N_i. Then we have $\overline{N}_1^j \cdot \overline{N}_2^{n+1-j} = 0$ for any j by $f^* \overline{N}_i = 4 \overline{N}_i$. It follows that

$$(\overline{N}_1 - \overline{N}_2)^2 \cdot (\overline{N}_1 + \overline{N}_2)^{n-1} = 0.$$

We are close to the setting of the theorem with $\overline{M} = \overline{N}_1 - \overline{N}_2$ and $\overline{L}_i = \overline{N}_1 + \overline{N}_2$. We can even adjust N_1 by a multiple to make

$$(N_1 - N_2) \cdot (N_1 + N_2)^{n-1} = 0.$$

Note that N_1, N_2 are not equal, we cannot apply the equality part of the theorem. The problem is that $\overline{N}_1 + \overline{N}_2$ is not ample.

1.2 Calabi–Yau theorem

When the line bundle M is trivial, Theorem 1.3 is essentially a result on projective varieties over local fields. It gives some cases of the non-archimedean Calabi–Yau Theorem. In fact, the proof generalizes to the following full generality.

Let K be either $\mathbb{C}$ or a field with a non-trivial complete non-archimedean absolute value $|\cdot|$. Let X be a geometrically connected projective variety over K, and L be an ample line bundle on X endowed with a continuous semipositive K-metric $\|\cdot\|$. Then $(L, \|\cdot\|)$ induces a canonical semipositive measure $c_1(L, \|\cdot\|)^{\dim X}$ on the analytic space X^{an}. We explain it as follows.

If $K = \mathbb{C}$, then X^{an} is just the complex analytic space $X(\mathbb{C})$. The measure $c_1(L, \|\cdot\|)^{\dim X}$ is just the determinant of the Chern form $c_1(L, \|\cdot\|)$ which is locally defined by

$$c_1(L, \|\cdot\|) = \frac{\partial\bar{\partial}}{\pi i} \log \|\cdot\|$$

in complex analysis.

If K is non-archimedean, X^{an} is the Berkovich space associated to the variety X over K. It is a Hausdorff, compact and path-connected topological space. Furthermore, it naturally includes the set of closed points of X. We refer to [Be] for more details on the space. The metric $\|\cdot\|$ being semipositive, in the sense of [Zh3], means that it is a uniform limit of metrics induced by ample integral models of L. The canonical measure $c_1(L, \|\cdot\|)^{\dim X}$ is constructed by Chambert-Loir [Ch] in the case that K contains a countable and dense subfield, and extended to the general case by Gubler [Gu2].

Theorem 1.5. *Let L be an ample line bundle over X, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on L. Then*

$$c_1(L, \|\cdot\|_1)^{\dim X} = c_1(L, \|\cdot\|_2)^{\dim X}$$

if and only if $\dfrac{\|\cdot\|_1}{\|\cdot\|_2}$ is a constant.

The history of the theorem in the complex case is as follows. In the 1950s, Calabi [Ca1, Ca2] made the following famous conjecture: *Let X be a compact complex manifold endowed with a Kahler form ω, and let Ω be a positive smooth volume form on X such that $\int_X \Omega = \int_X \omega^{\dim X}$. Then there exists a smooth real-valued function ϕ on X such that $(\omega + i\partial\overline{\partial}\phi)^{\dim X} = \Omega$.* Calabi proved that the function ϕ is unique up to scalars (if it exists). The existence of the function is much deeper, and was finally solved by S. T. Yau in the seminal paper [Ya] in 1977. Now the whole results are called the Calabi–Yau theorem.

If ω is cohomologically equivalent to a line bundle L on X, the results can be stated in terms of existence and uniqueness of metrics $\|\cdot\|$ on L with $c_1(L, \|\cdot\|)^{\dim X} = \Omega$.

Theorem 1.5 includes the non-archimedean analogue of the uniqueness part of the Calabi–Yau theorem. The "if" part of the theorem is trivial by definition. For archimedean K, the positive smooth case is due to Calabi as we mentioned above, and the continuous semipositive case is due to Kolodziej [Ko]. Afterwards Blocki [Bl] provided a very simple proof of Kolodziej's result.

The theorem will be proved analogously to the case of trivial M of Theorem 1.3. It can be viewed as a local arithmetic Hodge index theorem. Both theorems are proved utilizing Blocki's idea.

1.3 Algebraic dynamics

Let X be a projective variety over a field K. A *polarizable algebraic dynamical system on X* is a morphism $f : X \to X$ such that there is an ample $\mathbb{Q}$-line bundle $L \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $f^*L = qL$ from some rational number $q > 1$. We call L a *polarization of f*, and call the triple (X, f, L) a *polarized algebraic dynamical system*. Let $\mathrm{Prep}(f)$ denote the set of *preperiodic points*, i.e.,

$$\mathrm{Prep}(f) := \{x \in X(\overline{K}) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{N},\ m \neq n\}.$$

A well-known result of Fakhruddin [Fak] asserts that $\mathrm{Prep}(f)$ is Zariski dense in X.

Denote by $\mathcal{DS}(X)$ the set of all polarizable algebraic dynamical systems f on X. Note that we do *not* require elements of $\mathcal{DS}(X)$ to be polarizable by the same ample line bundle or have the same dynamical degree q.

Theorem 1.6. *Let X be a projective variety over a number field K. For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*;*

(2) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$*;*

(3) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in X.*

The theorem will be proved over any field K in our second paper [YZ]. The proof of the number field case in this paper gives a rough idea of our approach for the general case.

In an early version of the series, we require f and g to be polarizable by the same ample line bundle. The proof combines the equidistribution theorem of Yuan [Yu] generalizing that of Szpiro–Ullmo–Zhang [SUZ] et al, and the Calabi–Yau theorem in Theorem 1.5. But we have removed this restriction in the current version by introducing the arithmetic Hodge index theorem, which is more powerful than the Calabi–Yau theorem.

Remark 1.7. When $X = \mathbb{P}^1$, the theorem is independently proved by M. Baker and L. DeMarco [BD] over any field K of characteristic zero during the preparation of this paper. Their proof also applies to positive characteristics. Their treatment for the number field case is the same as our treatment in the earlier version, while the method for the general case is quite different.

Remark 1.8. The following are some works related to the above theorem (when K is a number field):

(1) A. Mimar [Mi] proved the theorem in the case that $X = \mathbb{P}^1$.

(2) If f is a Lattès map on $\mathbb{P}^1$ or a power map on $\mathbb{P}^d$ induced by $(\mathbb{G}_m)^d$, the theorem is implied by the explicit description of g by S. Kawaguchi and J. H. Silverman [KS].

(3) In the case that $X = \mathbb{P}^1$, C. Petsche, L. Szpiro, and T. Tucker [PST] found a further equivalent statement in terms of heights and intersections.

Now we explain some ingredients in our proof of Theorem 1.6. The hard part is to show that (3) implies (1).

Let $f, g \in \mathcal{DS}(X)$. We first consider the case that f and g are polarizable by the same ample line bundle L. Then we have an f-invariant adelic line

bundle $\overline{L}_f$ and a g-invariant adelic line bundle $\overline{L}_g$. Apply the successive minima to the sum $\overline{L}_f + \overline{L}_g$. By the assumption that $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense, the essential minimum of $\overline{L}_f + \overline{L}_g$ is 0. It follows that $(\overline{L}_f + \overline{L}_g)^{n+1} = 0$. By the nefness, it gives $\overline{L}_f^i \cdot \overline{L}_g^{n+1-i} = 0$ for any i. Hence, we have

$$(\overline{L}_f - \overline{L}_g)^2 \cdot (\overline{L}_f + \overline{L}_g)^{n-1} = 0.$$

Then we apply our arithmetic Hodge index theorem to conclude that $\overline{L}_f = \overline{L}_g$. It follows that f and g have the same canonical height. Then they have the same set of preperiodic points.

The general case that f and g are not polarizable by the same ample line bundle is much more difficult. We develop a theory of admissible adelic line bundles to overcome the difficulty. The idea is to write every line bundle as a sum of eigenvectors of f^*. For any ample class $\xi \in \mathrm{NS}(X)_{\mathbb{Q}}$, we first prove that there is a unique "f-admissible lifting" $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ of ξ. Note that L does not necessarily polarize f. Then we construct an f-admissible adelic line bundle $\overline{L}_f$ extending L. We prove that $\overline{L}_f$ is nef. Similarly, from ξ, we have a unique "g-admissible lifting" $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$ of ξ, and a unique g-admissible adelic line bundle $\overline{M}_g$ extending M. Here we a priori do not have $L = M$. But we can still apply the arithmetic Hodge index theorem by the above argument to force $\overline{L}_f = \overline{M}_g$.

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2 Arithmetic Hodge index theorem

The goal of this section is to prove Theorem 1.3. After introducing some basic positivity notions, we prove the theorem by a few steps which are clear from the section titles.

2.1 Terminology on arithmetic positivity

Let K be a number field. By an *arithmetic variety* $\mathcal{X}$ over O_K, we mean an integral scheme $\mathcal{X}$, projective and flat over O_K.

Recall that a *Hermitian line bundle* on $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a metric $\|\cdot\|$ on $\mathcal{L}(\mathbb{C})$ invariant under the action of the complex conjugation. The metric is assumed to satisfy the regularity that, for any analytic map from any complex open ball B (of any dimension) to $\mathcal{X}(\mathbb{C})$, the pull-back of $\|\cdot\|$ gives a smooth Hermitian metric on the pull-back of $\mathcal{L}(\mathbb{C})$ to B. We say that the metric is *semipositive* if any such pull-back has a positive semi-definite Chern form.

Definition 2.1. *(1) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is* nef *if the Hermitian metric $\|\cdot\|$ is semipositive, and the intersection $(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} \geq 0$ for any closed subvariety $\mathcal{Y}$ of $\mathcal{X}$.*

(2) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is ample *if the generic fiber $\mathcal{L}_K$ is ample on X, $\overline{\mathcal{L}}$ is nef, and the intersection $(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0$ for any horizontal closed subvariety $\mathcal{Y}$ of $\mathcal{X}$.*

(3) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is vertical *if the generic fiber $\mathcal{L}_K$ is trivial on X.*

Denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$ the group of Hermitian bundles on $\mathcal{X}$. Denote by $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{nef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{amp}}$, $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{vert}}$) the set of nef (resp. ample, vertical) Hermitian line bundles on $\mathcal{X}$.

Let X be a projective variety over K. By an *integral model $\mathcal{X}$ of X over O_K*, we mean an arithmetic variety $\mathcal{X}$ over O_K with a fixed isomorphism $\mathcal{X}_K = X$. Let L be a line bundle on X. If $\overline{\mathcal{L}}$ is a Hermitian line bundle on $\mathcal{X}$ with generic fiber $\mathcal{L}_K = L$, then we say that $\overline{\mathcal{L}}$ is an *integral model* of L on $\mathcal{X}$. We also say that $(\mathcal{X}, \overline{\mathcal{L}})$ is an *integral model* of (X, L) over O_K.

An *adelic line bundle on X* is a pair $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ consisting of a line bundle L on X and a collection of K_v-metrics $\|\cdot\|_v$ of $L(\overline{K}_v)$ on $X(\overline{K}_v)$ over

all places v of K. The K_v-metrics is assumed to be continuous and Galois invariant, and the collection is assumed to be coherent in that there is an integral model of (X, L) inducing a collection of K_v-metrics on $L(\overline{K}_v)$ which agrees with $\{\|\cdot\|_v\}_v$ for all but finitely many places v. We refer to [Zh2] for more details on the definition. The following is a copy of Definition 1.2, except that (5) is new.

Definition 2.2. *Let $\overline{L}, \overline{M}$ be adelic line bundles on X. We make the following definitions.*

(1) We say that $\overline{L}$ is nef *if it the adelic metric of L is a uniform limit of metrics induced by nef Hermitian line bundles on integral models of X.*

(2) We say that $\overline{L}$ is integrable *if it is the difference of two nef adelic line bundles on X.*

(3) We say that $\overline{L}$ is ample *if L is ample, $\overline{L}$ is nef, and $(\overline{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X.*

(4) We say that $\overline{M}$ is $\overline{L}$-bounded *if there is an integer $m > 0$ such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are nef.*

(5) We say that $\overline{L}$ is vertical *if L is trivial on X.*

Denote by $\widehat{\mathrm{Pic}}(X)$ the group of adelic line bundles on X. By the above definitions, $\widehat{\mathrm{Pic}}(X)$ has subsets $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X)_{\mathrm{amp}}$, $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ and $\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}}$.

All the positivity notions can be extended to the concepts of $\mathbb{Q}$-line bundles and $\mathbb{R}$-line bundles. For example, the space of integrable $\mathbb{Q}$-line bundles form a vector space $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

In the end, we present a lemma which shows the strength of the arithmetic ampleness defined above.

Lemma 2.3. *Let $\overline{L}$ be an adelic line bundle on X. Then the following are equivalent:*

(1) $\overline{L}$ is ample, i.e., L is ample, $\overline{L}$ is nef, and $(\overline{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X.

(2) L is ample, and $\overline{L} - \pi^*\overline{N}$ is nef for some adelic line bundle $\overline{N}$ on Spec K *with $\widehat{\deg}(\overline{N}) > 0$.*

Proof. It is easy to see that (2) implies (1). The implication from (1) to (2) follows the idea of the successive minima of [Zh2]. Assume (1) in the following. We claim that the absolute minima

$$\lambda(\overline{L}) = \inf_{x \in X(\overline{K})} h_{\overline{L}}(x)$$

is strictly positive. Apply the arithmetic Hilbert–Samuel formula for adelic line bundles in [Zh3, Theorem 1.7], which extends the original formula of Gillet–Soulé [GS2]. Combine it with the adelic Minkowski theorem. Replacing $\overline{L}$ by a multiple if necessary, we can find a nonzero section $s \in H^0(X, L)$ such that $\|s\|_{\mathbb{A}} < 1$. Here

$$\|s\|_{\mathbb{A}} = \prod_v \|s\|_{v,\sup}, \quad \|s\|_{v,\sup} = \sup_{x \in X(\overline{K}_v)} \|s(x)\|_v.$$

For any closed point $x \in X(\overline{K})$ such that $s(x) \neq 0$, use s to compute the height of x. It gives

$$\widehat{\deg}(\overline{L}|_x) \geq -\log \|s\|_{\mathbb{A}} > 0.$$

To show $\lambda > 0$, it suffices to consider algebraic points of the support of $\mathrm{div}(s)$. It has a smaller dimension, and can be done by induction.

Once we have $\lambda > 0$, the proof is immediate. In fact, we simply have

$$\lambda(\overline{L} - \pi^*\overline{N}) = \lambda(\overline{L}) - \widehat{\deg}(\overline{N}) > 0$$

for any adelic line bundle $\overline{N}$ on $\operatorname{Spec} K$ with $\widehat{\deg}(\overline{N}) < \lambda$. In this case, $\overline{L} - \pi^*\overline{N}$ is nef. □

Under some regularity assumption on the metric at infinity, the arithmetic Nakai–Moishezon criterion of Zhang [Zh3] holds for an ample adelic bundle $\overline{L}$. In particular, the regularity assumption is satisfied automatically if X is smooth over K.

2.2 Vertical case

We prove Theorem 1.3 assuming that M is trivial on X.

The inequality

To apply the full strength of intersection theory, we need some regularity property on arithmetic varieties. We say that an arithmetic variety is *vertically factorial* if all irreducible components of its special fibers are Cartier divisors. Let $\mathcal{X}$ be any arithmetic variety. After blowing up the irreducible components of the special fibers of $\mathcal{X}$ which are not Carter divisors successively (in any order), we end up with a vertically factorial arithmetic variety $\mathcal{X}'$ which dominates $\mathcal{X}$.

Proposition 2.4. *Let $\pi : \mathcal{X} \to \operatorname{Spec} O_K$ be an arithmetic variety of relative dimension n. Then the following are true:*

(1) If $\overline{\mathcal{M}} \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{vert}}$ and $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1} \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{nef}}$, then

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n-1} \leq 0.$$

(2) If $\overline{\mathcal{M}}_1, \overline{\mathcal{M}}_2 \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{vert}}$ and $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1} \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{nef}}$, then

$$(\overline{\mathcal{M}}_1 \cdot \overline{\mathcal{M}}_2 \cdot \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n-1})^2 \leq (\overline{\mathcal{M}}_1^2 \cdot \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n-1})(\overline{\mathcal{M}}_2^2 \cdot \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n-1}).$$

(3) Assume that $\mathcal{X}$ is vertically factorial. Let $\overline{\mathcal{M}} \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{vert}}$ and $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1} \in \widehat{\operatorname{Pic}}(\mathcal{X})_{\mathrm{nef}}$. Assume that $\mathcal{L}_i$ is ample on $\mathcal{X}$ and that the Hermitian metric of $\overline{\mathcal{L}}_i$ is strictly positive on the smooth locus of $\mathcal{X}(\mathbb{C})$ for every i. Then

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n-1} = 0 \iff \overline{\mathcal{M}} \in \pi^* \widehat{\operatorname{Pic}}(O_K).$$

Proof. The results are well-known for $n = 1$ and more or less for general n. The proof for general n is not more difficult than the case $n = 1$. We include it here for convenience.

Note that (2) is the Cauchy–Schwartz inequality induced by (1). In fact, let $x, y \in \mathbb{R}$ be variables. The quadratic form

$$\begin{aligned} & (x\overline{\mathcal{M}}_1 + y\overline{\mathcal{M}}_2)^2 \\ = \ & x^2\, \overline{\mathcal{M}}_1^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} + 2xy\, \overline{\mathcal{M}}_1 \cdot \overline{\mathcal{M}}_2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} + y^2\, \overline{\mathcal{M}}_2^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \end{aligned}$$

is negative semi-definite. It follows that the discriminant is negative or zero, which gives the inequality.

To prove (1), we can assume that $\mathcal{X}$ is also generically smooth and vertically factorial. In fact, let $\mathcal{X}'$ be the successive blowing-up of the non-Cartier irreducible components of the special fiber of the generic desingularization of $\mathcal{X}$ as above. Replace $\mathcal{X}$ by $\mathcal{X}'$ and all the Hermitian line bundles by the pull-backs on $\mathcal{X}'$.

Let $\overline{D} = (D, g)$ be a vertical arithmetic divisor representing $\overline{\mathcal{M}}$. Here the Green's function g is a continuous function on $\mathcal{X}(\mathbb{C})$. By definition,

$$\mathcal{M}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = \sum_{v \nmid \infty} D_v^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} + \int_{\mathcal{X}(\mathbb{C})} g \frac{\partial\overline{\partial}}{\pi i} g \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

Here D_v denotes the part of D above v. By integration by parts, the integral becomes

$$-\frac{1}{\pi i} \int_{\mathcal{X}(\mathbb{C})} \partial g \wedge \overline{\partial} g \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) \leq 0.$$

To show $D_v^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \leq 0$, enumerate the irreducible components of the special fiber $\mathcal{X}_v$ above v by $V_1, \cdots, V_r$. They are Cartier divisors by our assumption. We have $\mathcal{X}_v = \sum_{i=1}^r a_i V_i$ with multiplicity $a_i > 0$. For convenience, denote $E_i = a_i V_i$. Write $D_v = \sum_{i=1}^r b_i E_i$ with some $b_i \in \mathbb{R}$. We have

$$D_v^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \sum_{i,j=1}^r b_i b_j E_i \cdot E_j \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1}.$$

Note that

$$\sum_{j=1}^r b_i^2 \; E_i \cdot E_j \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = 0, \quad \forall i.$$

We obtain

$$\begin{aligned} D_v^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} &= -\frac{1}{2} \sum_{i,j=1}^r (b_i - b_j)^2 E_i \cdot E_j \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \\ &= -\frac{1}{2} \sum_{i \neq j} (b_i - b_j)^2 E_i \cdot E_j \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \\ &\leq 0. \end{aligned}$$

It finishes (1).

As for (3), we look at the conditions for the equality at every place. The integrals equal zero if and only if g is constant on every connected component of $\mathcal{X}(\mathbb{C})$. As for the intersection number above v, we have

$$E_i \cdot E_j \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} > 0, \qquad i \neq j$$

as long as $V_i \cap V_j$ is nonempty. For such i, j, we have $b_i = b_j$. It gives the equality of all b_i since the whole special fiber $\mathcal{X}_v$ is connected. $\square$

By taking limit, we have the following consequence. It gives the inequality of Theorem 1.3 in the vertical case.

Proposition 2.5. *Let $\pi : X \to \operatorname{Spec} K$ be a projective variety of dimension n. Then the following are true:*

(1) If $\overline{M} \in \widehat{\operatorname{Pic}}(X)_{\mathrm{vert}}$ and $\overline{L}_1, \cdots, \overline{L}_{n-1} \in \widehat{\operatorname{Pic}}(X)_{\mathrm{nef}}$, then

$$\overline{M}^2 \cdot \overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n-1} \leq 0.$$

(2) If $\overline{M}_1, \overline{M}_2 \in \widehat{\operatorname{Pic}}(X)_{\mathrm{vert}}$ and $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1} \in \widehat{\operatorname{Pic}}(X)_{\mathrm{nef}}$, then

$$(\overline{M}_1 \cdot \overline{M}_2 \cdot \overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n-1})^2 \leq (\overline{M}_1^2 \cdot \overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n-1})(\overline{M}_2^2 \cdot \overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n-1}).$$

Variational method

The key to obtain the equality part of Theorem 1.3 is the following result, which was inspired by [Bl].

Lemma 2.6. *Let $\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ be integrable adelic line bundles on X such that the following conditions hold:*

(1) M is trivial on X;

(2) $\overline{M}$ is $\overline{L}_i$-bounded for every i;

(3) $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0$.

Then for any nef adelic line bundles $\overline{L}_i^0$ on X with underlying bundle $L_i^0 = L_i$, and any integrable adelic line bundle $\overline{M}'$ with trivial underlying line bundle M', we have

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0.$$

Proof. Note that (2) implies every $\overline{L}_i$ is nef. Replacing with $\overline{L}_i$ by a large multiple if necessary, we can assume that both $\overline{L}_i \pm \overline{M}$ are nef for each i. Denote $\overline{L}_i^{\pm} = \overline{L}_i \pm \overline{M}$ in the following.

First, we have the equality

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0$$

for any sign function $\epsilon : \{1, \cdots, n-1\} \to \{+, -\}$. In fact, we can find a constant $t > 0$ such that $L_i - tL_i^{\epsilon(i)}$ is nef for any i. Then Proposition 2.5 gives

$$0 = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq \overline{M}^2 \cdot t\overline{L}_1^{\epsilon(1)} \cdots t\overline{L}_{n-1}^{\epsilon(n-1)} \leq 0.$$

It forces

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

Second, we claim that

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^{\epsilon(k)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0, \quad \forall\ k = 1, \cdots, n.$$

When $k = n$, then we can apply Proposition 2.5 (2) to conclude the proof.

Prove the claim by induction. We already have the case $k = 1$. Assume the equality for a general k for all sign functions ϵ. Apply Proposition 2.5 (2) to the vertical adelic line bundles $\overline{M}$ and $\overline{M}_k^{\epsilon(k)} = \overline{L}_k^{\epsilon(k)} - \overline{L}_k^0$, we have

$$\overline{M} \cdot \overline{M}_k^{\epsilon(k)} \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^{\epsilon(k)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

Replace $\overline{L}_k^{\epsilon(k)}$ by $\overline{L}_k^{+}$ and $\overline{L}_k^{-}$, and take the difference. We have

$$\overline{M} \cdot \overline{M}_k^{\epsilon(k)} \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{M} \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is just

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot (\overline{L}_k^{\epsilon(k)} - \overline{L}_k^0) \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

The left-hand side splits to a difference of two terms. One term is zero by the induction assumption for k. It follows that the other term is also zero, which gives

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^0 \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is exactly the case $k + 1$. The proof is complete. □

Condition of equality

To reduce adelic metrics to model metrics, we first introduce the "push-forward" of adelic line bundle to integral models.

Let $\overline{M}$ be an integrable adelic line bundle on $\mathcal{X}$ with $M = \mathcal{O}_X$ trivial. For any place v of K, the function $\log \|1\|_v$ on $X(\overline{K}_v)$ coming from the metric of $\overline{M}$ extends to a continuous function on the analytic space $X^{\mathrm{an}}_{K_v}$. By definition, $\log \|1\|_v = 0$ for all but finitely many v.

Let $\mathcal{X}$ be a vertically factorial integral model of X over O_K. Define a vertical arithmetic $\mathbb{R}$-divisor $\overline{D}_{\mathcal{X}}$ on $\mathcal{X}$ by

$$\overline{D}_{\mathcal{X}} := (-\sum_{(V,v)} \log \|1\|_v(\eta_V),\ -\log \|1\|_\infty).$$

Here the summation is over all pairs (V, v), where v is a non-archimedean place of K, and V is an irreducible component of the fiber $\mathcal{X}_v$ of $\mathcal{X}$ above v. The point $\eta_V \in X^{\mathrm{an}}_{K_v}$ denotes the Shilov point corresponding to V, which is the unique preimage of the generic point of V under the reduction map $X^{\mathrm{an}}_{K_v} \to \mathcal{X}_v$.

To get a line bundle, we denote by $\overline{M}_{\mathcal{X}} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{R}}$ the Hermitian line bundle on $\mathcal{X}$ associated to $\overline{D}_{\mathcal{X}}$. We have the following basic result.

Lemma 2.7. *(1) For any integrable Hermitian line bundles $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ on $\mathcal{X}$, we have*

$$\overline{M} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n = \overline{M}_{\mathcal{X}} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n.$$

(2) There is a sequence $\{\mathcal{X}_m\}_m$ of vertically factorial integral model of X over O_K such that the adelic metric induced by the push-forward $\overline{M}_{\mathcal{X}_m}$ converges uniformly to the adelic metric of $\overline{M}$.

Proof. By definition, there is a sequence of vertically factorial integral models $(\mathcal{X}_m, \overline{\mathcal{M}}_m)$ of (X, M) over O_K, which induces a sequence of adelic line bundles $\overline{M}_m$ convergent to $\overline{M}$. Here by abuse of terminology, convergence means uniform convergence of induced adelic metrics.

For (2), we prove that $\overline{M}_{\mathcal{X}_m}$ converges to $\overline{M}$. It suffices to show that $\overline{N}_m = \overline{M}_{\mathcal{X}_m} - \overline{M}_m$ converges to the trivial adelic line bundle. Let $\overline{E} = (E, g_E)$ be the vertical arithmetic divisor associated to the section 1 of $\overline{N}_m$. By definition, the function g_E on $X(\mathbb{C})$ and the multiplicities of irreducible components of special fibers of $\mathcal{X}$ in E converges uniformly to 0. It follows that $\overline{N}_m$ converges to the trivial adelic line bundle.

For (1), we can further assume that each $\mathcal{X}_m$ dominates $\mathcal{X}$ by a birational morphism $\alpha_m : \mathcal{X}_m \to \mathcal{X}$. By definition, $\overline{M}_{m,\mathcal{X}} = (\alpha_m)_* \overline{\mathcal{M}}_m$ converges to $\overline{M}_{\mathcal{X}}$, which can be understood as the convergence of the corresponding vertical divisors. Then (1) is just the limit of the projection formula

$$\overline{M}_m \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n = \overline{M}_{m,\mathcal{X}} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_n.$$

$\square$

Now we are ready to prove the equality part of Theorem 1.3 in the vertical case. Let $\overline{M}$ and $\overline{L}_1, \cdots \overline{L}_{n-1}$ be as in the equality part of theorem. Assume that M is trivial.

Let $\mathcal{X}$ be a vertically factorial integral model of X. For each m, let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{nef}}$ such that $\mathcal{L}_i$ is ample on $\mathcal{X}$ and that the Hermitian metric of $\overline{\mathcal{L}}_i$ is strictly positive on the smooth locus of $\mathcal{X}(\mathbb{C})$ for every i. By Lemma 2.6,

$$\overline{M} \cdot \overline{M}_{\mathcal{X}} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = 0.$$

By Lemma 2.7 (1), it becomes

$$\overline{M}_{\mathcal{X}} \cdot \overline{M}_{\mathcal{X}} \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = 0.$$

By Proposition 2.4 (3), $\overline{M}_{\mathcal{X}} \in \pi^* \widehat{\mathrm{Pic}}(O_K)$ where $\pi : \mathcal{X} \to \mathrm{Spec}\, O_K$ denotes the structure morphism by abuse of notation.

By Lemma 2.7 (2), the metric of $\overline{M}$ can be approximated uniformly by the metric induced by $\overline{M}_{\mathcal{X}}$ when varying $\mathcal{X}$. It follows that the limit $\overline{M}$ actually lies in $\pi^* \widehat{\mathrm{Pic}}(K)$. It finishes the proof of Theorem 1.3.

2.3 Case of curves

When $\dim X = 1$, Theorem 1.3 can be easily reduced to the result of Faltings [Fal] and Hriljac [Hr] on integral models. Note that $\overline{L}_i$ does not show up in the theorem.

We can assume that X is smooth by a desingularization. Take any integral model $\mathcal{X}$ of X over O_K. Then we can find some $\overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ extending M whose intersection with any vertical arithmetic divisor on $\mathcal{X}$ is 0. Denote by $\overline{M}_0 \in \widehat{\mathrm{Pic}}(X)$ the adelic line bundle induced by $\overline{\mathcal{M}}$. Define a vertical adelic line bundle $\overline{N}$ by

$$\overline{M} = \overline{M}_0 + \overline{N}.$$

By definition, $\overline{M}_0 \cdot \overline{N} = 0$ since $\overline{M}_0$ is perpendicular to all vertical classes. It follows that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2 \leq 0.$$

Here $\overline{M}_0^2 = -2\,\widehat{h}(M) \leq 0$ by the positivity of the Neron–Tate height, and $\overline{N}^2 \leq 0$ follows from the vertical case we treated before. The equality is attained if and only if $\overline{M}$ is torsion in $\widehat{\mathrm{Pic}}(X)/\pi^*\widehat{\mathrm{Pic}}(K)$ and and $\overline{N} \in \pi^*\widehat{\mathrm{Pic}}(K)$. The result is proved.

2.4 Inequality in the general case

Now we prove the inequality of Theorem 1.3 in the general case by induction on $n = \dim X$. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following.

Reducing to the model case

Consider the inequality part of the theorem. Recall $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ and $\overline{L}_1, \cdots, \overline{L}_{n-1} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$. The theorem assumes $M \cdot L_1 \cdots L_{n-1} = 0$ and that each L_i is big on X. We are going to prove

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

By a resolution of singularities, we can assume that X is smooth. Furthermore, we claim that we can further assume that each $\overline{L}_i$ is ample.

In fact, fix an ample adelic line bundle $\overline{A}$ on X. Take a small rational number $\epsilon > 0$. Set $\overline{L}_i' = \overline{L}_i + \epsilon\overline{A}$ and $\overline{M}' = \overline{M} + \delta\overline{A}$. Here δ is a number such that

$$M' \cdot L_1' \cdots L_{n-1}' = (M + \delta A) \cdot L_1' \cdots L_{n-1}' = 0.$$

It determines

$$\delta = -\frac{M \cdot L_1' \cdots L_{n-1}'}{A \cdot L_1' \cdots L_{n-1}'}.$$

As $\epsilon \to 0$, we have $\delta \to 0$ since

$$M \cdot L_1' \cdots L_{n-1}' \to M \cdot L_1 \cdots L_{n-1} = 0,$$

$$A \cdot L_1' \cdots L_{n-1}' \to A \cdot L_1 \cdots L_{n-1} > 0.$$

The last inequality uses the assumption that L_i is big and nef for each i. Therefore, the inequality $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0$ is the limit of the inequality $\overline{M}'^2 \cdot \overline{L}'_1 \cdots \overline{L}'_{n-1} \leq 0$. Here every $\overline{L}'_i$ is ample. So the claim is achieved.

Go back to the inequality

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

We have the condition $M \cdot L_1 \cdots L_{n-1} = 0$. We further assume that X is smooth, and $\overline{L}_i$ is ample for $i = 1, \cdots, n-1$. By approximation, it suffices to prove

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \leq 0$$

under the following assumptions:

- $\mathcal{X}$ is a normal integral model of X over O_K;
- $\overline{\mathcal{M}}, \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n-1}$ are Hermitian line bundles on $\mathcal{X}$ with smooth Hermitian metrics and with generic fiber $M, L_1, \cdots, L_{n-1}$;
- $\overline{\mathcal{L}}_i$ ample on $\mathcal{X}$ with strictly positive metric on $\mathcal{X}(\mathbb{C})$ for each $i = 1, \cdots, n-1$.

It was proved by Moriwaki [Mo1] in the case that all $\overline{\mathcal{L}}_i$ are equal. The current case is similar, but we still sketch it in the following.

The model case

Here we prove the inequality in the above model case.

First, by $M \cdot L_1 \cdots L_{n-1} = 0$, there is a metric $\|\cdot\|_0$ on $\mathcal{M}$, unique up to scalars, such that the Chern form of $\overline{\mathcal{M}}' = (\mathcal{M}, \|\cdot\|_0)$ gives

$$c_1(\overline{\mathcal{M}}')c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) = 0$$

pointwise on $X(\mathbb{C})$. It is given by an elliptic equation. We refer to [Gr, Corollary 2.2 A2] for a proof. Write the original metric of $\overline{\mathcal{M}}$ as $e^{-\phi}\|\cdot\|_0$ for a real-valued smooth function ϕ on $X(\mathbb{C})$. Then we have

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} - \int_{X(\mathbb{C})} \phi \frac{1}{\pi i} \partial\overline{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

By integration by parts, the second term on the right

$$-\int_{X(\mathbb{C})} \phi \frac{1}{\pi i} \partial\overline{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) = \int_{X(\mathbb{C})} \frac{1}{\pi i} \partial\phi \wedge \overline{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

It is non-positive since the volume from

$$\frac{1}{\pi i}\partial\phi \wedge \overline{\partial}\phi \wedge c_1(\overline{\mathcal{L}}_1)\cdots c_1(\overline{\mathcal{L}}_{n-1}) \leq 0.$$

Hence, it suffices to prove

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \leq 0.$$

By Moriwaki's arithmetic Bertini theorem in [Mo1], replacing $\overline{\mathcal{L}}_{n-1}$ by a tensor power if necessary, there is a nonzero section $s \in H^0(\mathcal{X}, \mathcal{L}_{n-1})$ satisfying the following conditions:

- The supremum norm $\|s\|_{\sup} = \sup_{x\in X(\mathbb{C})} \|s(x)\| < 1$;
- The horizontal part of $\mathrm{div}(s)$ on $\mathcal{X}$ is a generically smooth arithmetic variety $\mathcal{Y}$;
- The vertical part of $\mathrm{div}(s)$ on $\mathcal{X}$ is a linear combination $\sum_\wp a_\wp \mathcal{X}_\wp$ of smooth fibers $\mathcal{X}_\wp$ of $\mathcal{X}$ above (good) prime ideals $\wp$ of O_K.

Then by the intersection formula,

$$\begin{aligned} & \overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \overline{\mathcal{L}}_{n-1} \\ = \; & \overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{Y} + \sum_\wp a_\wp \, \overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{X}_\wp \\ & - \int_{X(\mathbb{C})} \log \|s\| c_1(\overline{\mathcal{M}})^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}). \end{aligned}$$

It suffices to prove each term on the right-hand side is non-positive. The "main term"

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{Y} \leq 0$$

by induction hypothesis. The vertical part

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{X}_\wp = M^2 \cdot L_1 \cdots L_{n-2} \log N_\wp \leq 0$$

follows from the geometric Hodge index theorem on the algebraic variety X (or $\mathcal{X}_\wp$). It remains to check

$$-\int_{X(\mathbb{C})} \log \|s\| c_1(\overline{\mathcal{M}})^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}) \leq 0.$$

In fact, we have a pointwise inequality

$$c_1(\overline{\mathcal{M}})^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}) \leq 0.$$

See [Gr, Lemma 2.1A]. It is called Aleksandrov's lemma, since it was essentially due to Aleksandrov [Al] assuming the existence of the metric $\|\cdot\|_0$.

2.5 Equality in the general case

Now we prove the second part of Theorem 1.3 in the general case. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following.

Argument on the generic fiber

Assume the conditions in the equality part of the Theorem 1.3, which particularly includes

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0.$$

We first show that M is numerically trivial on X by the condition that $\overline{L}_{n-1}$ is ample.

By Lemma 2.3, $\overline{L}'_{n-1} = \overline{L}_{n-1} - \pi^*\overline{N}$ is nef for some $\overline{N} \in \widehat{\mathrm{Pic}}(K)$ with $c = \widehat{\deg}(\overline{N}) > 0$. Then

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} + c\, M^2 \cdot L_1 \cdots L_{n-2}.$$

Applying the inequality of the theorem to $(\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-2}, \overline{L}'_{n-1})$, we have

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} \leq 0.$$

By the Hodge index theorem on X in the geometric case, we have

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

Hence,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} = M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

On the variety X, we have

$$M \cdot L_1 \cdots L_{n-2} \cdot L_{n-1} = 0, \quad M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

By the Hodge index theorem on algebraic varieties, we conclude that M is numerically trivial. See Theorem A.1 in the appendix.

Numerically trivial case

We have proved that M is numerically trivial on X. Here we continue to prove that M is a torsion line bundle. Then a multiple of $\overline{M}$ lies in the vertical case, which has already been treated.

As in the vertical case, the key is still the variational method.

Lemma 2.8. *Let $\overline{M}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ be integrable adelic line bundles on X such that the following conditions hold:*

(1) M is numeriacally trivial on X;

(2) $\overline{M}$ is $\overline{L}_i$-bounded for every i;

(3) $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0$.

For any nef adelic line bundles $\overline{L}_i^0$ on X with underlying bundle L_i^0 numerically equivalent to L_i, and any integrable adelic line bundle $\overline{M}'$ with numerically trivial underlying line bundle M', the following are true:

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0,$$

$$\overline{M}^2 \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-2}^0 = 0.$$

Proof. The first equality can be proved exactly in the same way as in Lemma 2.6. The second equality follows from the first one by vary $\overline{L}_{n-1}^0$. □

Go back to the equality part of Theorem 1.3. Apply Bertini's theorem. Replacing $\overline{L}_{n-1}$ by a positive multiple if necessary, there is a section $s \in H^0(X, L_{n-1})$ such that $Y = \mathrm{div}(s)$ is a normal subvariety of X. Then we have

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y.$$

In fact, the difference of two sides is the limit of the intersection of $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2}$ with vertical classes, so it vanishes by the second equality of Lemma 2.8. Hence,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y = 0.$$

By Theorem A.2, we can assume that $\mathrm{Pic}^0(X)_{\mathbb{Q}} \to \mathrm{Pic}^0(Y)_{\mathbb{Q}}$ is injective. Note that some multiple of M lies in $\mathrm{Pic}^0(X)$. It reduces the problem to Y. The proof is complete since we have already treated the case of curves.

2.6 Calabi–Yau theorem

The goal of this section is to treat Theorem 1.5 in the non-archimedean case. If (X, L) comes from a number field, it is essentially Theorem 1.3, the arithmetic Hodge index theorem for vertical adelic line bundles.

Let K be a non-archimedean field, X be a projective variety over K, and L be a line bundle on X. Recall that a *K-metric* on L is a continuous and $\mathrm{Gal}(\overline{K}/K)$-invariant collection of $\overline{K}$-metric $\|\cdot\|$ on $L(x)$ indexed by $x \in X(\overline{K})$. A *metrized line bundle* $\overline{L}$ on X is a pair $(L, \|\cdot\|)$ consisting of a line bundle L on X and a K-metric $\|\cdot\|$ on L. Denote by $\widehat{\mathrm{Pic}}(X)$ by the group of all metrized line bundles on X.

The metric is said to be *semipositive* if it is a uniform limit of metrics induced by integral models $(\mathcal{X}_m, \mathcal{L}_m)$ of (X, L) over O_K where $\mathcal{L}_m \in \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ is nef on on fibers of $\mathcal{X}_m$ above O_K. The metric is said to be *integrable* if it is the quotient of two semipositive metrics. By abuse of notations, we also say that the corresponding metrized line bundle are semipositive or integrable if the metric is so. To be compatible with the global case, semipositive metrized line bundles are also called *nef metrized line bundles*. Finally, we also have the notion of $\overline{L}$-bounded as in the global base.

Theorem 2.9 (local hodge index theorem)**.** *Let K be a non-archimedean field, and $\pi : X \to \mathrm{Spec}\, K$ be a geometrically connected projective variety of dimension $n \geq 1$. Let $\overline{M}$ be an integrable metrized line bundle on X with M trivial, and $\overline{L}_1, \cdots, \overline{L}_{n-1}$ be $n-1$ nef metrized line bundles on X. Then*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if L_i is ample and $\overline{M}$ is $\overline{L}_i$-bounded for each i, then the equality holds if and only if $\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)$.

The theorem is a local version of Theorem 1.3 in the vertical case. If K is the completion of a number field K_0 at some place, and (X, L) is the base changes to K of a pair (X_0, L_0) over K_0. Then Theorem 2.9 is equivalent to Theorem 1.3.

The proof of Theorem 1.3 in the vertical case is easily translated to a proof of Theorem 2.9. We omit it. However, it is worth noting that, for a general non-archimedean field K, integral models of X over O_K are not Noetherian, so the usual intersection theory is not applicable. In that case, Gubler [Gu1] introduced an intersection theory using rigid-analytic geometry, and the translation goes through by his intersection theory.

Now we go back to Theorem 1.5. We will prove the following stronger result as in the case of model metrics.

Theorem 2.10. *Let L be an ample line bundle over X, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on L. View $f = -\log(\|\cdot\|_1/\|\cdot\|_2)$ as a continuous function on X^{an}. Then*

$$\int_{X^{\mathrm{an}}} f \ c_1(L, \|\cdot\|_1)^{\dim X} = \int_{X^{\mathrm{an}}} f \ c_1(L, \|\cdot\|_2)^{\dim X}$$

if and only if f is a constant.

This theorem is just a simple consequence of Theorem 2.9. In fact, denote $\overline{L}_1 = (L, \|\cdot\|_1)$ and $\overline{L}_2 = (L, \|\cdot\|_2)$. Then the equality of the integrals is just

$$(\overline{L}_1 - \overline{L}_2) \cdot \overline{L}_1^n = (\overline{L}_1 - \overline{L}_2) \cdot \overline{L}_2^n.$$

Here $n = \dim X$. Equivalently,

$$\sum_{i=0}^{n-1} (\overline{L}_1 - \overline{L}_2)^2 \cdot \overline{L}_1^i \cdot \overline{L}_2^{n-1-i} = 0.$$

By Theorem 2.9, every term in the sum is non-positive. It forces

$$(\overline{L}_1 - \overline{L}_2)^2 \cdot \overline{L}_1^i \cdot \overline{L}_2^{n-1-i} = 0, \quad \forall\ i = 0, \cdot, n-1.$$

It follows that

$$(\overline{L}_1 - \overline{L}_2)^2 \cdot (\overline{L}_1 + \overline{L}_2)^{n-1} = 0.$$

Note that $\overline{L}_1 - \overline{L}_2$ is vertical and $(\overline{L}_1 + \overline{L}_2)$-bounded. Theorem 2.9 implies $\overline{L}_1 - \overline{L}_2 \in \pi^*\widehat{\mathrm{Pic}}(K)$, which is equivalent to the statement that f is a constant.

3 Algebraic dynamics

In this section, we prove Theorem 1.6. We first introduce a theory of admissible adelic line bundles which will be needed in the proof. Then we prove the theorem using our arithmetic Hodge index theorem.

3.1 Admissible arithmetic classes

Let (X, f, L) be a *polarized dynamical system* over a number field K, i.e.,

- X is a projective variety over K;
- $f : X \to X$ is a morphism over K;
- $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is an ample line bundle such that $f^*L = qL$ from some $q > 1$.

By [Zh3], Tate's limiting argument gives an adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ extending L and with $f^*\overline{L}_f = q\overline{L}_f$. In the following we generalize the definition to construct an admissible metric on any line bundle $M \in \mathrm{Pic}(X)$.

Semisimplicity

By definition, f^* preserves the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

It is known that $\mathrm{NS}(X)$ is a finitely generated $\mathbb{Z}$-module. Assume that X is normal. Then $\mathrm{Pic}^0(X)$ is also a finitely generated $\mathbb{Z}$-module. In fact, the Picard functor $\underline{\mathrm{Pic}}^0_{X/K}$ is represented by an abelian variety A. See [Kl, Theorem 5.4] for example. Then $\mathrm{Pic}^0(X) = A(K)$ is just the Mordell–Weil group. Alternatively, one can obtain the finiteness using only the Picard variety for a resolution of singularities.

Theorem 3.1. *Let (X, f, L) be a polarized dynamical system over a number field K. Assume that X is normal.*

(1) The operator f^* is semisimple on $\mathrm{Pic}^0(X)_{\mathbb{C}}$ (resp. $\mathrm{NS}(X)_{\mathbb{C}}$) with eigenvalues of absolute values $q^{1/2}$ (resp. q).

(2) the operator f^* is semisimple on $\mathrm{Pic}(X)_{\mathbb{C}}$ with eigenvalues of absolute values $q^{1/2}$ or q.

Proof. The result can be proved using Hodge–Riemann bilinear relation (for Betti cohomology). See Serre [Ser] for the case when X is smooth. Here we present an algebraic proof which can be generalized to positive characteristics. It suffices to prove (1), since (2) is a consequence of (1).

We first consider $\mathrm{NS}(X)_{\mathbb{C}}$. Write $n = \dim X$ as usual. Make the decomposition

$$\mathrm{NS}(X)_{\mathbb{R}} := \mathbb{R}L \oplus P(X), \quad P(X) = \{\xi \in \mathrm{NS}(X)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\}.$$

By Theorem A.1, the pairing

$$\langle \xi_1, \xi_2 \rangle = \xi_1 \cdot \xi_2 \cdot L^{n-2}$$

is a negative definite quadratic form on $P(X)$. The projection formula gives

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle.$$

If follows that $q^{-1}f^*$ is an orthogonal transformation (with respect to the quadratic form). Then $q^{-1}f^*$ is diagonalizable with eigenvalues of absolute values 1.

Next we consider $\mathrm{Pic}^0(X)_{\mathbb{C}}$. For $\xi_1, \xi_2 \in \mathrm{Pic}^0(X)$, define a pairing

$$(\xi_1, \xi_2) = \overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}.$$

Here $\overline{\xi}_i$ is an adelic line bundle on X extending ξ_i and with zero intersection with any vertical classes on X for $i = 1, 2$, and $\overline{L}$ is any adelic line bundle on X extending L. The intersection does not depend on the choice of the extension $\overline{L}$ since $\overline{\xi}_1$ is perpendicular to any vertical class.

The extension $\overline{\xi}_i$ always exists. In fact, we already know that $A = \underline{\mathrm{Pic}}^0_{X/K}$ is an abelian variety. Let P be the universal bundle on $X \times A$. Then for any $\alpha \in A(\overline{K})$, the line bundle $P|_{X\times\alpha}$ on X is exactly the element of $\underline{\mathrm{Pic}}^0_{X/K}(\overline{K})$ represented by α. Rigidify P by $P|_{x_0\times A} = 0$ for some point $x_0 \in X(K)$. Here we assume x_0 exists by extending K if necessary. The multiplication $[2]_X : X \times A \to X \times A$ on the second component gives $[2]_X^* P = 2P$. By Tate's limiting argument, we obtain an adelic line bundle $\overline{P}$ extending P with $[2]_X^*\overline{P} = 2\overline{P}$. Then $\overline{P}|_{X\times\alpha_i}$ gives the desired extension of ξ_i, where $\alpha_i \in A(K)$ is the point representing ξ_i.

Go back to the pairing

$$(\xi_1, \xi_2) = \overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}.$$

It is extended to a pairing on $\mathrm{Pic}^0(X)_{\mathbb{R}}$. We claim that the pairing is negative definite. In fact, let C be a closed non-singular curve in X representing L^{n-1}, which exists by Bertini's theorem. Then we have

$$(\xi_1, \xi_2) = \overline{\xi}_1|_C \cdot \overline{\xi}_2|_C = -2\langle \xi_1|_C,\ \xi_2|_C\rangle_{\mathrm{NT}}.$$

Here we used the Hodge index theorem of [Fal, Hr]. By Theorem A.2, the map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(C)$ has a finite kernel. It follows that the paring is negative definite.

On the other hand, the projection formula gives

$$(f^*\xi_1, f^*\xi_2) = q\ (\xi_1, \xi_2).$$

It follows that $q^{-1/2}f^*$ is an orthogonal transformation (with respect to the pairing). Then $q^{-1/2}f^*$ is diagonalizable with eigenvalues of absolute values 1. The result is proved.

□

By the theorem above, the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}} \longrightarrow \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow 0.$$

has a splitting

$$\ell_f : \mathrm{NS}(X)_{\mathbb{C}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{C}}$$

by identifying $\mathrm{NS}(X)_{\mathbb{C}}$ with the subspace of $\mathrm{Pic}(X)_{\mathbb{C}}$ generated by eigenvectors belonging to eigenvalues of absolute values q. It is easy to see that the splitting actually descends to

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}.$$

Definition 3.2. We say an element of $\mathrm{Pic}(X)_{\mathbb{C}}$ is *f-pure of weight* 1 *(resp.* 2*)* if it lies in $\mathrm{Pic}^0(X)_{\mathbb{C}}$ (resp. $\ell_f(\mathrm{NS}(X)_{\mathbb{C}})$).

Admissible metrics

Now we introduce f-admissible metrics for line bundles. Note that the eigenvalues can be imaginary numbers. So we first extend the group of adelic line bundles.

Recall that $\widehat{\mathrm{Pic}}(X)$ is the group of adelic line bundles on X. Assume that $\pi : X \to \operatorname{Spec} K$ is geometrically connected. Write $\mathbf{F}$ for $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Similar to [Mo4], we introduce

$$\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]} := \frac{\widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbf{F}}{(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F})_0}.$$

Here we describe the subspace in the denominator. Denote

$$C(X, \mathbf{F}) = \oplus_v C(X_{K_v}^{\mathrm{an}}, \mathbf{F}),$$

where $C(X_{K_v}^{\mathrm{an}}, \mathbf{F})$ denotes the space of continuous functions from $X_{K_v}^{\mathrm{an}}$ to $\mathbf{F}$. When $\mathbf{F} = \mathbb{Z}, \mathbb{Q}$, it is endowed with the discrete topology, so $C(X_{K_v}^{\mathrm{an}}, \mathbf{F}) = \mathbf{F}$ in these cases. The map $\log \|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \to C(X, \mathbb{R})$ extends to an $\mathbf{F}$-linear map

$$\log \|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F} \to C(X, \mathbf{F}).$$

Define

$$(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F})_0 := \ker(\log \|1\| : \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F} \to C(X, \mathbf{F})).$$

By definition, it is easy to have

$$\widehat{\mathrm{Pic}}(X)_{[\mathbb{Z}]} = \widehat{\mathrm{Pic}}(X), \quad \widehat{\mathrm{Pic}}(X)_{[\mathbb{Q}]} = \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}.$$

However, they are not true for $\mathbb{R}$ or $\mathbb{C}$. But we still have

$$\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]} = \widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]} \otimes_{\mathbb{R}} \mathbb{C}.$$

Define $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbf{F}]}$ to be the image of $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbf{F}$ in $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$. The intersection theory extends to $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbf{F}]}$ by linearity. The positivity notions are extended to $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},[\mathbb{R}]}$.

The action $f^* : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(X)$ extends to $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ naturally. The goal is to study the spectral theory of this action.

Definition 3.3. An element $\overline{M}$ of $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$ is called *f-admissible* if we can write $\overline{M} = \sum_{i=1}^m \overline{M}_i$ such that each $\overline{M}_i$ is an eigenvector of f^* in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$.

The main result here asserts the existence of an admissible section of the forgetful map $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]} \to \mathrm{Pic}(X)_{\mathbb{C}}$.

Theorem 3.4. *For any $M \in \mathrm{Pic}(X)_{\mathbb{C}}$, there exists a unique f-admissible lifting $\overline{M}_f$ of M in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$. Moreover, for $\mathbf{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, if $M \in \mathrm{Pic}(X)_{\mathbf{F}}$, then $\overline{M}_f \in \widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$.*

Proof. It suffices to assume that $M \in \mathrm{Pic}(X)_{\mathbb{C}}$ is an eigenvector of f^*. Namely, $f^*M = \lambda M$ with $|\lambda| = q^{1/2}$ or q. We claim that Tate's limiting argument of [Zh3] still works here. Namely, let $\overline{M}$ be any lifting of M in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$, then $(\lambda^{-1}f^*)^m\overline{M}$ converges *uniformly* to a unique element in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$. This limit does not depend on the choice of $\overline{M}$, and we define $\overline{M}_f$ to be this limit. We only sketch an idea of the proof.

Let $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ be an element represented by

$$\overline{M} = \sum_{i=1}^{r} a_i \otimes \overline{M}_i, \quad a_i \in \mathbf{F},\ \overline{M}_i \in \widehat{\mathrm{Pic}}(X).$$

Take *an* $\mathbf{F}$*-section* s of $\overline{M}$, i.e., a formal product

$$s = \otimes_{i=1}^{r} s_i^{\otimes a_i}, \quad s_i \in H^0(X, M) \setminus \{0\}.$$

It has *divisor*

$$\mathrm{div}(s) = \sum_{i=1}^{r} a_i\ \mathrm{div}(s_i) \in \mathrm{Div}(X)_{\mathbf{F}}.$$

Then the metrics turn to a function

$$-\log\|s\| = -\sum_{i=1}^{r} a_i \log\|s_i\|$$

in the space

$$S(X, \mathbf{F}) = \oplus_v S(X_{K_v}^{\mathrm{an}}, \mathbf{F}),$$

where $S(X_{K_v}^{\mathrm{an}}, \mathbf{F})$ denotes the space of functions from $X_{K_v}^{\mathrm{an}}$ to $\mathbf{F}$ with logarithmic singularity along some $\mathbf{F}$-divisor. Then we convert the uniform convergence of $\overline{M}$ to the uniform convergence of $-\log\|s\|$.

With these preparations, the original argument works here. We remark that we can further define the equivalence of two $\mathbf{F}$-sections s_1 and s_2 by $\mathrm{div}(s_1) = \mathrm{div}(s_2)$. $\square$

Example 3.5. If $M \in \mathrm{Pic}^0(X)$, then $\overline{M}_f$ is represented by an arithmetic class which is perpendicular to all vertical arithmetic classes. The arithmetic class was used in the proof of Theorem 3.1.

Example 3.6. The admissible metrics for abelian varieties are well-known. Let X be an abelian variety and $f = [m]$ be the multiplication by m on X. Here m is an integer with $|m| > 1$. Any symmetric and ample line bundle L on X gives a polarization of (X, f). We have $q = m^2$ in this case. Then $\ell_f(\mathrm{NS}(X)_{\mathbb{Q}})$ consists of exactly the rational multiples of symmetric line bundles, and $\mathrm{Pic}^0(X)$ is exactly the group of anti-symmetric line bundles. It is easy to see that f^* acts as m on $\mathrm{Pic}^0(X)$, and as m^2 on $\ell_f(\mathrm{NS}(X)_{\mathbb{Q}})$. The f-admissible classes can be obtained by the usual Tate's limiting argument (without get $\mathrm{Pic}(X)_{\mathbb{C}}$ involved).

Recall that we have a canonical section

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$$

for the surjection $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{NS}(X)_{\mathbb{Q}}$. By the f-admissible classes, we get a section

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X)_{[\mathbb{Q}]}$$

for the surjection $\widehat{\mathrm{Pic}}(X)_{[\mathbb{Q}]} \to \mathrm{NS}(X)_{\mathbb{Q}}$.

Definition 3.7. For $\mathbf{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, define

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbf{F}} \longrightarrow \widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$$

to be the map which sends $\xi \in \mathrm{NS}(X)_{\mathbf{F}}$ to the unique f-admissible class in $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ extending $\ell_f(\xi)$.

Positivity

The key result for our application is the following assertion.

Theorem 3.8. *Let (X, f, L) be a polarized dynamical system over a number field K with X normal. If $M \in \mathrm{Pic}(X)_{\mathbb{R}}$ is ample and f-pure of weight 2, then $\overline{M}_f$ is nef.*

Proof. By the action of the complex conjugation, we have $\overline{M}_f \in \widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$.

Let $\overline{M}$ be any nef extension of M in $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$. Still consider the sequence $\overline{M}_m = (q^{-1}f^*)^m\overline{M}$. Here every term $\overline{M}_m$ is nef. We will pick a subsequence "convergent" to $\overline{M}_f$. Decompose

$$\overline{M} = \sum_{i=1}^{r} \overline{N}_i,$$

where $f^*N_i = \lambda_i N_i$ with $|\lambda_i| = q$ for any i. Then

$$\overline{M}_m = \sum_{i=1}^r (q^{-1}f^*)^m \overline{N}_i.$$

To compare with

$$\overline{M}_f = \sum_{i=1}^r \overline{N}_{i,f},$$

write

$$\overline{M}_m = \sum_{i=1}^r (q^{-1}\lambda_i)^m \cdot (\lambda_i^{-1}f^*)^m \overline{N}_i.$$

Here $(\lambda_i^{-1}f^*)^m \overline{N}_i$ converges to $\overline{N}_{i,f}$ uniformly.

Since $|q^{-1}\lambda_i| = 1$, we can find an infinite subsequence $\{m_k\}_k$ such that $(q^{-1}\lambda_i)^{m_k} \to 1$ for every i. Then $\overline{M}_{m_k}$ "converges" to $\overline{M}$ in the sense of combining the uniform convergence of adelic metrics and the convergence of coefficients.

Then the theorem follows from the lemma below. Note that $\overline{N}_{i,f}$ and $(q^{-1}f^*)^m \overline{N}_i$ may lie in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{C}]}$ instead of in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$. But $\overline{M}, \overline{M}_f$ and $\overline{M}_m$ are in $\widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$, and thus we can take the real parts of their decompositions above to get only elements of $\widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$ involved. $\square$

Lemma 3.9. *Suppose we are given*

$$\overline{M} = \sum_{i=1}^r a_i \overline{N}_i, \quad \overline{M}_m = \sum_{i=1}^r a_{i,m} \overline{N}_{i,m}, \quad a_i \in \mathbb{R}, \quad a_{i,m} \in \mathbb{R}.$$

Here every $\overline{N}_i$ and every $\overline{N}_{i,m}$ are adelic line bundles on a projective variety X over a number field. For any $i = 1, \cdots, r$, assume the following convergence conditions:

- *$a_{i,m} \to a_i$ as $m \to \infty$;*
- *$N_{i,m} = N_i$ for any m;*
- *$\overline{N}_{i,m}$ converges uniformly to $\overline{N}_i$ as $m \to \infty$.*

If M is ample and $\overline{M}_m$ is nef for any m, then $\overline{M}$ is nef.

Proof. Let $\overline{M}^0$ and $\overline{N}_i^0$ be any adelic line bundles extending M and N_i, with some conditions we will impose later. Denote

$$\overline{M}_m^0 = \sum_{i=1}^r a_{i,m}\overline{N}_i^0.$$

Let $\{\epsilon_m\}_m$ be a sequence in the interval $(0,1)$ convergent to 0. Consider

$$\overline{M}_m' = (1-\epsilon_m)\overline{M}_m - (1-\epsilon_m)\overline{M}_m^0 + \overline{M}^0.$$

Note that

$$\overline{M}_m' - \overline{M} = (1-\epsilon_m)(\overline{M}_m - \overline{M}_m^0) + (\overline{M}^0 - \overline{M}).$$

We see that the underlying line bundle of $\overline{M}_m'$ is exactly M, and $\overline{M}_m'$ converges to $\overline{M}$ uniformly. We are going to pick $\overline{M}^0, \overline{N}_i^0$ and ϵ_m so that $\overline{M}_m'$ is nef, which will prove the lemma. The conditions assume that $\overline{M}_m$ is nef. It suffices to make

$$-(1-\epsilon_m)\overline{M}_m^0 + \overline{M}^0 = \epsilon_m\Big(\overline{M}^0 + (\epsilon_m^{-1}-1)(\overline{M}^0 - \overline{M}_m^0)\Big)$$

nef.

Let $\mathcal{X}$ be an integral model of X. Let $\overline{M}^0$ and $\overline{N}_i^0$ be induced by Hermitian line bundles $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}_i$ on $\mathcal{X}$. Denote

$$\overline{\mathcal{M}}_m = \sum_{i=1}^r a_{i,m}\overline{\mathcal{N}}_i.$$

Assume $\overline{\mathcal{M}}$ satisfies the following strong positivity conditions:

- $\overline{\mathcal{M}}$ is ample in the arithmetic sense;
- $\mathcal{M}$ is ample in the geometric sense;
- There is an embedding of $\mathcal{X}(\mathbb{C})$ to a compact complex manifold Ω, such that $\mathcal{M}(\mathbb{C})$ can be extended to an ample line bundle on Ω and the metric of $\overline{\mathcal{M}}$ can be extended to a positive (smooth) metric of the ample line bundle on Ω.

We further assume that the metric of $\overline{\mathcal{N}}_i$ satisfies the regularity condition:

- For the same complex manifold Ω as above, the line bundle $\overline{\mathcal{N}}_i(\mathbb{C})$ can be extended to a line bundle on Ω and the metric of $\overline{\mathcal{N}}_i$ can be extended to a smooth metric of the line bundle on Ω.

These assumptions make sure that, in the (finite-dimensional) real vector subspace V of $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{R}}$ generated by $\overline{\mathcal{N}}_1, \cdots, \overline{\mathcal{N}}_r$ and $\overline{\mathcal{M}}$, the subset of ample $\mathbb{R}$-line bundles in V form a neighborhood of $\overline{\mathcal{M}}$. Here V is endowed with the Euclidean topology. This is the open property of arithmetic ampleness and can be checked by the arithmetic Nakai–Moishezon criterion of Zhang [Zh2].

By definition, $\overline{\mathcal{M}} - \overline{\mathcal{M}}_m$ converges to 0 in V. Hence, as long as ϵ_m converges to 0 much more slowly, the line bundle

$$\overline{\mathcal{M}} + (\epsilon_m^{-1} - 1)(\overline{\mathcal{M}} - \overline{\mathcal{M}}_m)$$

is ample on $\mathcal{X}$. The proof is complete.

□

3.2 Preperiodic points

The goal of this section is to prove Theorem 1.6.

Canonical heights

Let (X, f, L) be a polarized dynamical system over a number field K with X normal. Let M be any line bundle on X. *The canonical height* $h_{\overline{M}_f} : X(\overline{K}) \to \mathbb{R}$ associated to M is defined by

$$h_{\overline{M}_f}(x) = \frac{1}{\deg(x)} \widehat{\deg}(\overline{M}_f|_{\tilde{x}}), \quad x \in X(\overline{K}).$$

Here $\tilde{x}$ denotes the closed point of X representing x. The following are some basic properties:

- For any line bundle M, $h_{\overline{M}_f}(x) = 0$ if x is preperiodic.
- If M is ample, then $h_{\overline{M}_f}$ satisfies the Northcott property.
- If M is ample and f-pure of weight 2, then $h_{\overline{M}_f}(x) \geq 0$ for any $x \in X(\overline{K})$, since $\overline{M}_f$ is nef.

- For the polarization line bundle L, $h_{\overline{L}_f}(x) = 0$ if and only if x is preperiodic. It is an old result.

More generally, we define *the canonical height of a closed subvariety* Y of X associated to M by

$$h_{\overline{M}_f}(Y) := \frac{1}{(\dim Y + 1)\deg_M(Y)}\overline{M}^{\dim Y+1}.$$

We still have the result that preperiodic subvarieties have canonical height 0. In fact, we have the following result.

Proposition 3.10. *Let Y be a preperiodic closed subvariety of X of dimension r. Let $M_1, M_2, \cdots, M_{r+1}$ be line bundles in $\mathrm{Pic}(X)_{\mathbb{C}}$ which are f-pure of weight two. Then we have*

$$\overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y = 0.$$

Proof. By the projection formula, we can assume that Y is periodic. Replacing f by a power if necessary, we can further assume that $f(Y) = Y$. By linearity, we can assume that M_i is an eigenvector of f^* for every i. Write $f^*M_i = \lambda_i M_i$. Then $f^*\overline{M}_{i,f} = \lambda_i \overline{M}_{i,f}$. Then the projection formula gives

$$f^*\overline{M}_{1,f} \cdot f^*\overline{M}_{2,f} \cdots f^*\overline{M}_{r+1,f} \cdot Y = q^r\ \overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y.$$

It follows that

$$(\lambda_1 \cdots \lambda_{r+1} - q^r)\ \overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y = 0.$$

The result follows since $\lambda_1 \cdots \lambda_{r+1} - q^r \neq 0$, as a consequence of the assumption $|\lambda_i| = q$. $\square$

Example 3.11. The result is not true without the assumption that the line bundles are f-pure of weight two. Take X to be an elliptic curve, and M_1, M_2 to be two line bundles of degree zero. Then they are f-pure of weight one if we take $f = [2]$. Then $\overline{M}_{1,f} \cdot \overline{M}_{2,f} = -2\langle M_1, M_2\rangle_{\mathrm{NT}}$ is often nonzero.

Preperiodic points

Now we are ready to prove the following theorem, which refines Theorem 1.6 in the case of number fields. The condition of X being normal can be obtained by taking a normalization.

Theorem 3.12. *Let X be a normal projective variety over a number field K. For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$;

(2) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$;

(3) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in* X;

(4) $\widehat{\ell}_f = \widehat{\ell}_g$ *as maps from* $\mathrm{NS}(X)_{\mathbb{Q}}$ *to* $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)$\Rightarrow$ (1). We start the proof with some easy directions.

First, (1) $\Rightarrow$ (2) it trivial.

Second, (2) $\Rightarrow$ (3). For any integer $d > 0$, denote

$$\mathrm{Prep}(f, d) := \{x \in \mathrm{Prep}(f) \mid \deg(x) < d\}.$$

By Northcott's property, $\mathrm{Prep}(f, d)$ is a finite set since its points have trivial canonical heights. Assuming (2), then g stabilizes the set $\mathrm{Prep}(f)$. By definition,

$$\mathrm{Prep}(f) = \bigcup_{d>0} \mathrm{Prep}(f, d).$$

Since g is also defined over K, it stabilizes the set $\mathrm{Prep}(f, d)$. By the finiteness, we obtain that

$$\mathrm{Prep}(f, d) \subset \mathrm{Prep}(g), \ \forall d.$$

Hence,

$$\mathrm{Prep}(f) \subset \mathrm{Prep}(g).$$

Then (3) is true since $\mathrm{Prep}(f)$ is Zariski dense in X by the result of Fakhruddin [Fak].

Third, we prove (4) $\Rightarrow$ (1). Let L be an ample line bundle on X polarizing f. By (4), $\overline{L}_f = \overline{L}_g$. For any $x \in \mathrm{Prep}(g)$, we have $h_{\overline{L}_f}(x) = h_{\overline{L}_g}(x) = 0$. It follows that $x \in \mathrm{Prep}(f)$. This proves $\mathrm{Prep}(g) \subset \mathrm{Prep}(f)$. By symmetry, we have the other direction and thus the equality.

From (3) to (4)

Assume that $\mathrm{Prep}(f)\cap\mathrm{Prep}(g)$ is Zariski dense in X. Write $n=\dim X$. We need to prove $\widehat{\ell}_f(\xi)=\widehat{\ell}_g(\xi)$ for any $\xi\in \mathrm{NS}(X)_{\mathbb{Q}}$. By linearity, it suffices to assume that ξ is ample. Denote $L=\ell_f(\xi)$ and $M=\ell_g(\xi)$. They are ample $\mathbb{Q}$-line bundles on X. Then $\overline{L}_f=\widehat{\ell}_f(\xi)$ and $\overline{M}_g=\widehat{\ell}_g(\xi)$ are nef by Theorem 3.8.

Consider the sum $\overline{N}=\overline{L}_f+\overline{M}_g$, which is still nef. By the successive minima of Zhang [Zh3],

$$\lambda_1(X,\overline{N})\geq h_{\overline{N}}(X)\geq 0.$$

Here

$$h_{\overline{N}}(X)=\frac{1}{(n+1)\deg_N(X)}\overline{N}^{n+1}$$

and the essential minimum

$$\lambda_1(X,\overline{N})=\sup_{U\subset X}\inf_{x\in U(\overline{K})}h_{\overline{N}}(x),$$

where the supremum is taken over all Zariski open subsets U of X. Note that $h_{\overline{N}}$ is zero on $\mathrm{Prep}(f)\cap\mathrm{Prep}(g)$, which is assumed to be Zariski dense in X. Hence, $\lambda_1(X,\overline{N})=0$. It forces $h_{\overline{N}}(X)=0$.

Write in terms of intersections, we have

$$(\overline{L}_f+\overline{M}_g)^{n+1}=0.$$

Expand by the binomial formula. Note that every term is non-negative. It follows that

$$\overline{L}_f^i\cdot\overline{M}_g^{n+1-i}=0,\quad \forall i=0,1,\cdots,n+1.$$

It particularly gives

$$(\overline{L}_f-\overline{M}_g)^2\cdot(\overline{L}_f+\overline{M}_g)^{n-1}=0.$$

Note that

$$(L-M)\cdot(L+M)^{n-1}=0$$

since $L-M\in\mathrm{Pic}^0(X)_{\mathbb{Q}}$ is numerically trivial. We are in the situation to apply Theorem 1.3 to

$$(\overline{L}_f-\overline{M}_g,\ \overline{L}_f+\overline{M}_g).$$

It is immediate that $(\overline{L}_f - \overline{M}_g)$ is $(\overline{L}_f + \overline{M}_g)$-bounded. The only condition that does not match the theorem is that $(\overline{L}_f + \overline{M}_g)$ is not ample. However, since $L - M$ is numerically trivial, as in the remark after the theorem, we can take any $\overline{C} \in \widehat{\mathrm{Pic}}(K)$ with $\deg(\overline{C}) > 0$, and replace

$$(\overline{L}_f - \overline{M}_g,\ \overline{L}_f + \overline{M}_g)$$

by

$$(\overline{L}_f - \overline{M}_g,\ \overline{L}_f + \overline{M}_g + \pi^*\overline{C}).$$

Then all the conditions are satisfied. The theorem implies that

$$\overline{L}_f - \overline{M}_g \in \pi^*\widehat{\mathrm{Pic}}(K).$$

By evaluating at any point x in $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$, we see that

$$\overline{L}_f - \overline{M}_g = 0$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. Here we have used the restriction $\overline{L}_f|_x = 0$, which is more delicate than $\widehat{\deg}(\overline{L}_f|_x) = 0$. It finishes the proof.

3.3 Variants and questions

Now we consider some variants, consequences and questions related to Theorem 1.6.

Variants

In a private communication, Barry Mazur points out that one direction of Theorem 1.6 can be generalized as follows:

Theorem 3.13. *Let X be a projective variety over a number field K. Let $f, g \in \mathcal{DS}(X)$, and denote by Y the Zariski closure of* $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *in X. Then*

$$\mathrm{Prep}(f) \cap Y(\overline{K}) = \mathrm{Prep}(g) \cap Y(\overline{K}).$$

The proof of Theorem 1.6 applies here. In fact, we can always restrict f-admissible (or g-admissible) adelic line bundles from X to Y. Then we apply Theorem 1.3 on Y.

One consequence of Theorem 3.12 is the following result.

Corollary 3.14. *Let X be a projective variety over a number field K, and $f, g \in \mathcal{DS}(X)$ be two polarizable algebraic dynamical systems. If* $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in X, then $d\mu_{f,v} = d\mu_{g,v}$ on $X_{K_v}^{\mathrm{an}}$ for any place v of K.*

Here $d\mu_{f,v}$ denotes the equilibrium measure of (X, f) on the analytic space $X_{K_v}^{\mathrm{an}}$. It can be obtained from any initial "smooth" measure on $X_{K_v}^{\mathrm{an}}$ by Tate's limiting argument. By a proper interpretation, it satisfies $f^* d\mu_{f,v} = q^{\dim X} d\mu_{f,v}$ and $f_* d\mu_{f,v} = d\mu_{f,v}$.

On can deduce the theorem by the equivalent condition $\widehat{\lambda}_f = \widehat{\lambda}_g$ in Theorem 3.12. Alternatively, one can apply the equidistribution theorem of Yuan [Yu] to any generic sequence in $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ to obtain the result.

Semigroup

For any subset P of $X(\overline{K})$, denote

$$\mathcal{DS}(X, P) := \{g \in \mathcal{DS}(X) \mid \mathrm{Prep}(g) = P\}.$$

We say that that P is a *special set of X* if $\mathcal{DS}(P)$ is non-empty.

Question 3.15. Let P be a special set of X. Is the set $\mathcal{DS}(X, P)$ is a semigroup?

The question asks whether $g \circ h \in \mathcal{DS}(X, P)$ for any $g, h \in \mathcal{DS}(X, P)$. By Theorem 1.6, we can write:

$$\mathcal{DS}(X, P) = \{g \in \mathcal{DS}(X) \mid gP \subset P\}.$$

Then $P \subset \mathrm{Prep}(g \circ h)$ by the simple argument proving (2) $\Rightarrow$ (3) of Theorem 1.6. Then we have $g \circ h \in \mathcal{DS}(X, P)$ if $g \circ h$ is polarizable.

The polarizability is automatically true if X is a projective space. Therefore, if $X = \mathbb{P}^n$, then $\mathcal{DS}(X, P)$ is a semigroup.

The results and the questions also apply to general fields by the treatment of [YZ].

Dynamical Manin–Mumford

The second author of this paper proposed in [Zh3] a dynamical analogue of the classical Manin–Mumford conjecture for abelian varieties. Namely, given

a dynamical triple (X, L, f) over a field K of characteristic zero, a closed subvariety Y of X is preperiodic under f if and only if the set $Y(\overline{K}) \cap \mathrm{Prep}(f)$ is Zariski dense in Y. Recently, Ghioca and Tucker found the following counter-example of this conjecture:

Proposition 3.16 (Ghocia and Tucker, [GTZ])**.** *Let E be an elliptic curve with complex multiplication by an order R in imaginary quadratic field K. Let f be an endomorphism on $E \times E$ defined by multiplications by two nonzero elements α and β in R with equal norm $\mathrm{N}(\alpha) = \mathrm{N}(\beta)$. Then f is polarized by any symmetric and ample line bundle*

$$\mathrm{Prep}(f) = E_{\mathrm{tor}} \times E_{\mathrm{tor}}.$$

Moreover, the diagonal Δ_E in $E \times E$ is not preperiodic under f if α/β is not a root of unity,

Notice that the diagonal is preperiodic for multiplication by $(2, 2)$. Thus the proposition shows *an example that two endomorphisms of a projective variety, with the same set of preperiodic points, have different sets of preperiodic subvarieties.* We would like to propose the following revision of the dynamical Manin–Mumford conjecture:

Question 3.17. Let X be a projective variety over any field K, and P be a special set of X. Let Y be a proper closed subvariety of X such that $Y(\overline{K}) \cap P$ is Zariski dense in Y. Do there exist two endomorphisms $f, g \in \mathcal{DS}(X, P)$, and a proper g-periodic closed subvariety Z such that $f(Y) \subset Z$?

If the answers to both questions are positive, then in the situation of the second question, we can find a finite sequence of subvarieties

$$X := X_0 \supset X_1 \supset X_2 \cdots \supset X_s$$

and endomorphisms $f_i, g_i \in \mathcal{DS}(X_i, P \cap X_i(\overline{K}))$ for $i = 0, 1, \cdots, s$ such that

(1) X_i is g_i-periodic, which implies that $P \cap X_i(\overline{K})$ is a special set of X_i;

(2) $f_i \circ f_{i-1} \cdots f_0(Y) \subset X_{i+1}$ for any $i = 0, 1, \cdots, s-1$;

(3) $f_{s-1} \circ f_{s-2} \cdots f_0(Y) = X_s$.

A Lefschetz theorems for Normal Varieties

We list some classical Lefschetz-type results applicable to normal projective varieties over any characteristic. Their counterparts for complex projective manifolds are even more classical, and we refer them to [La, Chapter 3].

Let X be a projective variety of dimension $n \geq 2$ over an algebraically closed field k. Consider the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

Here $\mathrm{Pic}^0(X)$ denotes the subgroup of algebraically trivial line bundles, and $\mathrm{NS}(X)$ denotes the quotient group.

Recall that a line bundle L on X is numerically trivial if $L \cdot C = 0$ for any closed curve C in X. It is well-known that a line bundle L is numerically trivial if and only if the multiple mL is algebraically trivial for some nonzero integer m. See [Kl, Theorem 6.3].

Theorem A.1. *Let $L_1, \cdots, L_{n-1}$ be ample line bundles on X. For any $M \in \mathrm{Pic}(X)$ with $M \cdot L_1 \cdots L_{n-1} = 0$, one has*

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

The equality holds if and only if M is numerically trivial.

Proof. If X is a smooth projective surface, the result is the classical Hodge index theorem. See [Ha, Theorem IV.1.9] for example. If X is a singular projective surface, the result is induced by a desingularization $X' \to X$. Note that the pull-back L_1' of L_1 to X' is not ample, but $L_1'^2 = L_1^2 > 0$ is sufficient for the result.

In general, by Bertini's theorem, we can assume that $L_1 \cdot L_2 \cdots L_{n-2}$ is represented by an integral closed surface S in X. Then the inequality is proved by the Hodge index theorem on S.

For the condition of the equality, we need to prove that $M \cdot C = 0$ for any complete curve C in X. By Bertini's theorem, we can assume that $L_1 \cdot L_2 \cdots L_{n-2}$ is represented by an effective 2-cycle $\sum_{i=0}^r a_i S_i$ with $a_i > 0$ such that S_0 contains C. Then

$$\sum_{i=0}^r a_i S_i \cdot M^2 = 0$$

implies that $S_i \cdot M^2 = 0$ since each term is non-positive. It follows that $M|_{S_0}$ is numerically trivial on S_0. Then $M \cdot C = 0$. $\square$

By a very ample linear system, we mean a subspace V of $H^0(X, L)$, for a very ample line bundle L on X, which gives an embedding $X \hookrightarrow \mathbb{P}(V)$. Denote by $|V| = \{\mathrm{div}(s) : s \in V\}$ the space of hyperplane sections. By a general hyperplane section of V in X, we mean an element in a Zariski open subset of $|V|$. By the Bertini-type result of Seidenberg [Sei], if X is normal and projective, then a general element $Y \in |V|$ is also normal and projective. The following is the Lefschetz hyperplane theorem in the current setting.

Theorem A.2. *Let X be a normal projective variety of dimension n over k. Let Y be a general hyperplane section of a very ample linear system V in X.*

(1) The natural map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ has a finite kernel if $n \geq 2$.

(2) The natural map $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ has a finite kernel if $n \geq 3$.

(3) The natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ has a finite kernel if $n \geq 3$.

Proof. Part (3) is a consequence of (1) and (2). For (1), we refer to [Kl, Remark 5.8] for a historical account. Part (2) is a consequence of Theorem A.1. In fact, assume that M lies in the kernel of $\mathrm{NS}(X) \to \mathrm{NS}(Y)$. In Theorem A.1, set $L_1 = \cdots = L_{n-1} = \mathcal{O}(Y)$. We see that M is numerically trivial on X. Then some integer multiple of M lies in $\mathrm{Pic}^0(X)$. Hence, the kernel of $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ is a torsion subgroup. It must be finite since $\mathrm{NS}(X)$ is a finitely generated abelian group. □

Remark A.3. The theorems remain true if X is projective and regular in codimension one, i.e., the singular locus X_{sing} has codimension at least 2.

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